Lecture 2

Wednesday, 26 August 2026 1:27 AM

Last week:

Cake-cutting: $C = [0,1]$, n agents, valuation fn. $v_i : \mathcal{B}[0,1] \to \mathbb{R}_+$

Assumed: non-atomic, non-negative, additive, normalized

EF, PROP

Robertson-Webb query model: $Cut_i(x, \alpha)$, $Eval_i(x, y)$

PROP for n agents, exact partition for 2 agents.

Now:

Dubins-Spanier moving knife procedure, for n agents gives a PROP allocation in $O(n^2)$ cut queries

Even-Paz does this in $O(n \log n)$ using divide & conquer.

Assume n is a power of 2.

$x \leftarrow 0$, $y \leftarrow 1$, $d \leftarrow 1$, $N' \leftarrow N$

Even-Paz (x, y, d, N')

If $|N'| = 1$, give $[x, y]$ to $i \in N'$.

Else let $x_i \leftarrow Cut_i(x, \frac{1}{2^d})$ $\forall i \in N'$ // assume all distinct

let $x^* \leftarrow \text{median}(\{x_i\}_{i \in N'})$

Let $N_L \leftarrow \{i \in N' : x_i \le x^*\}$, $N_R \leftarrow N' \setminus N_L$

Even-Paz $(x, x^*, d+1, N_L)$, Even-Paz $(x^*, y, d+1, N_R)$

Theorem: Even-Paz gives a connected PROP allocation in $O(n \log n)$ cut queries.

(prove yourself)

Can you do better? Yes and no.

Theorem: Any deterministic protocol requires $O(n \log n)$ queries in the RW model.

V. recently (Arndt et al., 2026):

Theorem: There is a deterministic algo that achieves PROP using $O(n)$ cut queries & $O(n^2)$ eval queries.

Envy-freeness.

Satisfies:

1. no interpersonal comparison. Don't compare an agent's utility with another.
2. scale-invariance
3. no agent trades places with another.

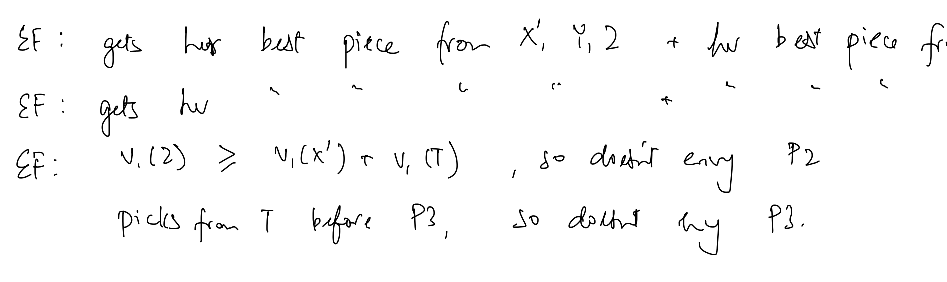
Q1. Does an EF allocation exist?

Q2. Can we find an EF allocation using a small # of RW queries?

Selfridge-Conway Procedure for 3 agents:

Generalizes cut-and-choose.

1. P1 cuts cake into X, Y, Z of equal value. $v_1(X) = v_1(Y) = v_1(Z)$
2. Assume $v_2(X) \geq v_2(Y) \geq v_2(Z)$.
3. If $v_2(X) = v_2(Y)$:
   P3 chooses a piece among X, Y, Z first, then P2, then P1.
   This is EF since everyone gets a best piece.
4. Else if $v_2(X) > v_2(Y) \geq v_2(Z)$:
5. P2 divides X into X', T so that $v_2(X') = v_2(Y) \geq v_2(Z)$.
   T is called the trimmings.
6. Now P3 chooses first among X', Y, Z.
   P2 takes X' if remaining, else chooses Y.
   P1 takes remaining piece among X', Y, Z. This is EF, since:
   P3 takes best piece, P2 takes best piece, P1 chooses b/w Y & Z, both of which are highest-value pieces.
   But we still need to allocate T.
7. Consider the case: P3 takes Y, P2 takes X', P1 takes Z.
8. Player that takes Y (In this case P3) partitions T into T1, T2, T3 s.t.
   $v_3(T1) = v_3(T2) = v_3(T3)$
   Then: P2 chooses her largest value piece (say T1)
   P1 chooses from T2 & T3
   P3 takes remaining piece.
   This is EF.



$v_1(X) = v_1(X') + v_1(T) = v_1(Y) = v_1(Z)$, $v_2(X) > v_2(Y) \geq v_2(Z)$
& $v_2(X') = v_2(Y)$

P3 is EF: gets her best piece from X', Y, Z + her best piece from T1, T2, T3

P2 is EF: gets her " " " " + " " " "

P1 is EF: $v_1(Z) \geq v_1(X') + v_1(T)$, so doesn't envy P2
picks from T before P3, so doesn't envy P3.

Note that, if envy-freeness is our only concern, there is a trivial soln.: no one gets anything, just throw away the entire cake.

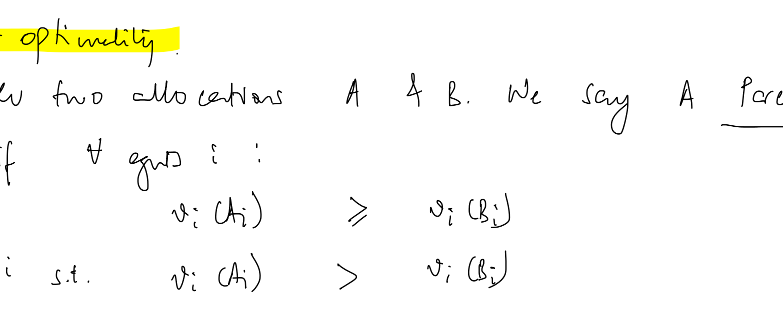
So fairness by itself may not be a satisfactory criterion.

Example: 2 agents,

Let density fn. $f_1(x) = 2$ for $x \in [0, 1/4] \cup [1/2, 3/4]$
$= 0$ o.w.

$f_2(x) = 2$ for $x \in [1/4, 1/2] \cup [3/4, 1]$
$= 0$ o.w.

& $Eval_1(x, y) = \int_x^y f_1(x)\,dx$, $Eval_2(x, y) = \int_x^y f_2(x)\,dx$



then the allocation $A_1 = [0, 1/2]$, $A_2 = [1/2, 1]$
is both EF & PROP, but is a bad allocation.

Pareto-optimality

Consider two allocations A & B. We say A Pareto dominates B if $\forall$ agents i:

$v_i(A_i) \geq v_i(B_i)$

& $\exists i$ s.t. $v_i(A_i) > v_i(B_i)$

An allocation A is Pareto optimal if no allocation Pareto dominates A.

Note that previous allocation is not PO.

Also a PO allocation may be very unfair.

PO also avoids interpersonal comparisons.

(BUT:

Theorem: There is no finite algorithm using RW queries that gives a PO allocation, even for 2 agents.

(Kurokawa, Lai, Procaccia, 2013)

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A PO characterization for 2 agents:

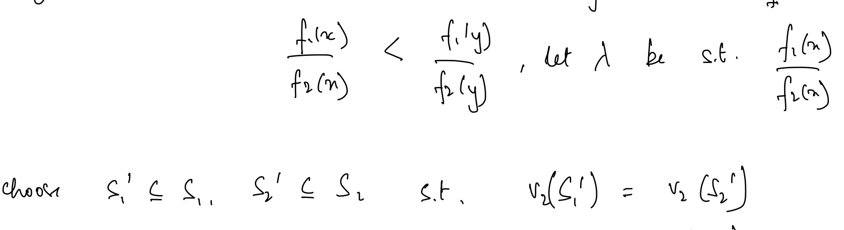
We assume agents have bounded valuation density fns. $f_1, f_2 : [0,1] \to \mathbb{R}_+$,
so that $v_i(S) = \int_{x \in S} f_i(x)\,dx$

Define the valuation density ratio $r(x) = \dfrac{f_1(x)}{f_2(x)}$
$= 0$ if $f_1(x) = f_2(x) = 0$
$= \infty$ if $f_2(x) = 0$

Theorem: An allocation $A = (A_1, A_2)$ is PO iff $\exists$ a threshold $\theta \in [0, \infty]$
s.t., up to sets of measure zero,
$\{x : r(x) > \theta\} \subseteq A_1$ & $\{x : r(x) < \theta\} \subseteq A_2$

Ties may be split arbitrarily.

Proof (only if): Suppose for a contradiction there is no such threshold.
Then there exist measurable sets $S_1 \subseteq A_1$, $S_2 \subseteq A_2$
s.t. $\forall x \in S_1$, $y \in S_2$, $r(x) < r(y)$.



$\dfrac{f_1(x)}{f_2(x)} < \dfrac{f_1(y)}{f_2(y)}$, let λ be s.t. $\dfrac{f_1(x)}{f_2(x)} < \lambda < \dfrac{f_1(y)}{f_2(y)}$

Now choose $S_1' \subseteq S_1$, $S_2' \subseteq S_2$ s.t. $v_2(S_1') = v_2(S_2')$

Then $\dfrac{v_1(S_1')}{v_2(S_1')} < \lambda$, $\dfrac{v_1(S_2')}{v_2(S_2')} > \lambda$, hence $v_1(S_2') > v_1(S_1')$

Swapping S_1', S_2' increases value for P1, keeps P2 unchanged.

('if' part left for later). ∎